Lecture 16: Cohomology of Projective Space

Note Title 11/3/2019

Theorem 1: X = (Noetherian) separated scheme

$\mathcal{F}$ = quasi-coherent $\mathcal{O}_X$-module

$\Rightarrow$ Čech cohomology w.r.t affine open cover = sheaf cohomology

Theorem 2: A = Noetherian ring, $S = k[x_0, \cdots, x_n]$, $n \geq 1$

$X = \operatorname{Proj} S = \mathbb{P}^n_A$

$\Longrightarrow$

① $S \cong \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(n))$

② $H^i(X, \mathcal{O}_X(l)) = 0$, $\forall\, 0 < i < n$, $l \in \mathbb{Z}$

③ $H^n(X, \mathcal{O}_X(-n-1)) \cong A$

④ $H^0(X, \mathcal{O}_X(l)) \times H^n(X, \mathcal{O}_X(-n-l-1)) \longrightarrow H^n(X, \mathcal{O}_X(-n-1)) \cong A$

perfect pairing of finitely generated free A-module, $\forall\, l \in \mathbb{Z}$

Theorem 3. (Serre) X: projective scheme over Noetherian ring A w/ $\mathcal{O}_X(1)$ very ample invertible sheaf

$\mathcal{F}$ = coherent sheaf on X

Then ① $H^i(X, \mathcal{F})$ finitely generated A-module

② $\exists\, n_0 \in \mathbb{N}$ s.t $H^i(X, \mathcal{F}(n)) = 0$, $\forall\, i > 0$, $n > n_0$

pf: • $X \xrightarrow{i} \mathbb{P}^N_A$

$$H^i(X, \mathcal{F}) = H^i(\mathbb{P}^n_A, i_* \mathcal{F})$$

$$i_*(\mathcal{F}(n)) = (i_* \mathcal{F})(n)$$

Thus, it suffices to prove the Theorem for $X = \mathbb{P}^n_A$.

• From Theorem 2, Theorem true for $\mathcal{F} = \mathcal{O}(\ell)$

• Recall that every coherent sheaf $\mathcal{F}$.

$$\exists\ \mathcal{E} = \bigoplus_{i=1}^{r=1} \mathcal{O}_X(n_i) \quad \text{s.t} \quad \mathcal{E} \twoheadrightarrow \mathcal{F}$$

$$0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0$$

$$\to \underbrace{H^i(X, \mathcal{E})}_{\substack{\text{finitely generated} \\ \text{by Theorem 2}}} \to H^i(X, \mathcal{F}) \to \underbrace{H^{i+1}(X, \mathcal{R})}_{\substack{\text{finitely generated} \\ \text{by descending induction on } i \\ H^{i>n}(X, \mathcal{R}) = 0}}$$

$$0 \to \mathcal{R}(\ell) \to \mathcal{E}(\ell) \to \mathcal{F}(\ell) \to 0, \quad \ell \gg 0$$

$$\underbrace{H^i(X, \mathcal{E}(\ell))}_{=0} \to H^i(X, \mathcal{F}(\ell)) \xrightarrow{\cong} \underbrace{H^{i+1}(X, \mathcal{R}(\ell))}_{=0} \to \underbrace{H^{i+1}(X, \mathcal{E}(\ell))}_{=0}$$

Theorem 2

by descending induction on i

$H^{i>n}(X, \mathcal{R}(\ell)) = 0$

Remark: ① is true if X is proper over A

use higher derived image of $\mathcal{F}$

Theorem 4 : A = Noetherian ring
X = proper scheme over A
$\mathcal{L}$ = invertible sheaf on X

TFAE : ① $\mathcal{L}$ is ample

② $\forall$ coherent sheaf $\mathcal{F}$ on X
$\exists\, n_0 = n_0(\mathcal{F})$ s.t $H^i(X, \mathcal{F}(n)) = 0,\ \forall i > 0,\ n \geq n_0$

pf : ① $\Rightarrow$ ②

$\mathcal{L}$ ample $\Longrightarrow$ $\mathcal{L}^{\otimes n}$ very ample

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr] & & \mathbb{P}^n_A \arrow[dl] \\ & \operatorname{Spec} A & \end{tikzcd}$$

$\therefore$ X projective $/A$

Theorem 3 $\Longrightarrow$ $\forall\, \mathcal{G}$ coherent sheaf
$\exists\, k_0 = k_0(\mathcal{G})$ s.t $H^i(X, \mathcal{G}(kn)) = 0,\ \forall i > 0,\ k \geq k_0$

Choose $\mathcal{G} = \mathcal{F}, \mathcal{F}(1), \cdots, \mathcal{F}(n-1)$

& $n_0 = n \cdot \max\{k_0(\mathcal{F}), k_0(\mathcal{F}(1)), \cdots, k_0(\mathcal{F}(n-1))\}$

② $\Rightarrow$ ①

want to prove that $\mathcal{F}(n)$ generated by global sections, $n \gg 0$

$\forall p \in X$, $0 \longrightarrow \mathcal{I}_p \otimes \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes k(p) \longrightarrow 0$ $\otimes \mathcal{L}^m$

$$\rightsquigarrow \quad 0 \to \mathcal{I}_p \otimes \mathcal{F}(n) \to \mathcal{F}(n) \to \mathcal{F}(n) \otimes k(p) \to 0$$

$$H^0(X, \mathcal{F}(n)) \twoheadrightarrow \mathcal{F}(n) \otimes k(p) \to H^1(X, \mathcal{I}_p \otimes \mathcal{F}(n)) = 0$$

② $n_1 = n_1(\mathcal{F}, p)$

$\therefore$ $\mathcal{F}(n)$ generated by global sections in a neighborhood by Nakayama's lemma.

Take $n_2 \in \mathbb{N}$ s.t $\mathcal{L}^{\otimes n_2}$ generated by global sections

Replace $\mathcal{F}$ by $\mathcal{F}(1), \ldots, \mathcal{F}(n_2 - 1)$